On Chrysanthos' Diatonic Scale

(Part One)

by

Ioannis Arvanitis

[One] way of dealing with errors is to have friends who are willing to spend the time necessary to carry out the critical examination of the experimental design beforehand and the results after the experiments have been completed. An even better way is to have an enemy. An enemy is willing to devote a vast amount of time and brain power to ferreting out errors both large and small, and this without compensation. The trouble is that really capable enemies are scarce; most of them are only ordinary. Another trouble with enemies is that they sometimes develop into friends and lose a good deal of their zeal. It was in this way that the writer lost his three best enemies.

[Georg von Békésy, quoted in: John Backus, *The Acoustical Foundations of Music,* W.W. Norton & Company, 2nd edition, 1977],

The purpose of this small article is to show that the concept of *komma* in the way it is introduced by Chrysanthos contradicts the very concept of komma as a *unit* of measurement of musical intervals of Byzantine Chant.

Intervals are traditionally expressed as ratios of integers a/b with $a<b$, where a/b (<1) is the fraction of a chord length representing the *sounding part* of the chord and the fret producing a note whose musical interval with respect to the note produced by the whole chord is the given interval. So, this fraction is a length measured from the big bridge, found on the soundboard, of an instrument like tanbur. The corresponding note (pitch) and interval *are not* the chord length fraction, as a length or pure number, *itself*; they are *represented* by it. Pitches can be said to constitute a one-dimensional abstract space, the 'pitch space'. A musical interval is the 'distance' between two points in the pitch space and is *represented* by a ratio in the ordinary space. A distance in the pitch space is not represented by the *difference* of the corresponding ratios but by their *quotient*. Addition of distances in the pitch space (=addition of musical intervals) is not represented by the *sum* but by the *product* of the corresponding ratios and subtraction not by the *difference* but by the *quotient*. The quest for kommata is the quest for a unit of measuring distances in the pitch space itself. The kommata as abstract units of measurement in the abstract pitch space were first axiomatically defined by Aristoxenus without any reference to the corresponding ratios and their algebra. But in the following paragraph, we can see that the kommata can be deduced from the ratios representing pitches and intervals.

A komma should be a musical interval, expressed through a real or, more precisely, an irrational (*pure*) number <1 (as the concept of komma includes in itself the concept of a kind of *synkrasis*, i.e. temperament, one cannot expect a komma to be expressed through a *ratio* of integers) or through its logarithm. In the first case and for an interval of n kommata, the kommata should be multiplied by themselves or, in other words, the ratio representing the komma should be raised to the n-th power, in order to produce, or actually to approximate, the given interval. So, to *define* a komma, one should solve the exponential equation: $k^m=a/b$, where k is the unknown real number for the komma and a/b is an interval which we want *by definition* to divide in m kommata. E.g: with $k^{72}=1/2$ we divide the octave in 72 kommata, or with $k'^{12}=8/9$ we divide the meizon tonos in 12 kommata. These kommata are equal pure numbers (<1), so acoustically equal musical intervals, and can be used to measure a given interval. Having defined the komma k , we can find the number of kommas constituting a certain interval by solving the equation: $k^x= M/N$, where M/N is the given interval. This equation is solved by taking the logarithms of both sides: *log*

$(k^{\chi})=log(M/N)$ or $\chi\ logk=log(M/N)$ and so $\chi=(1/logk)*log(M/N)$. Having defined the k, $1/logk$ is a fixed number, say A, and $\chi=A\ log(M/N)$. So, taking simply the log of the ratio representing a given interval and multiplying by A, we find the kommata of a given interval. Now if M/N is an interval of m kommata and M'/N' an interval of n kommata, the composite interval, i.e. the interval produced by the addition of them, is $(M/N)*(M'/N')$. But $M/N=k^m$ and $M'/N'=k^n$. So, $(M/N)*(M'/N')=k^m*k^n=k^{m+n}$. So, the composite interval has $m+n$ kommata. But in this way, we can see that we can use these numbers, the kommata, instead of using the ratios. It is more difficult to multiply ratios and it is more difficult to estimate the magnitude of an interval or compare two intervals when they are expressed as ratios. Kommata (defined in the above way) are acoustical units, they are simple, pure numbers and can be handled by the simple mathematical praxeis of addition and subtraction. Expressing intervals in kommata is a mere mathematical transformation through the logarithmic function which has the property of transforming products to sums. It is not something inferior or something destroying the concept of interval. It is a good and correct way of approximating them in a form suitable for immediate estimation and comparison.

But, as it will be shown, Chrysanthos' kommata are not pure numbers; they have the dimension of length. They are equal fractions of the chord length. As such they are geometrically but not acoustically equal. So, they are not acoustical *units* and if they are not units, they cannot be used to measure something and they cannot be added freely for acoustical purposes.

On page 26, note b), Chrysanthos writes:

"*That the intervals Di Ke, Ke Zo, Zo Ni have a ratio like 12, 9, 7 is demonstrated from this: Di Ke:Ke Zo::(1/9):(1/12) ie. (4:36):(3:36), but also as (4:36):12::(3:36):x, which is 4:(12*36)::3:(36x), so 4*36x=12*36*3, consequently x=9*".

But what are these "intervals"? Are they the acoustical intervals which are expressed through ratios of whole numbers? -Not, at all. Di Ke here is the *distance* from the small bridge of the tambur to the frets representing the note Ke. Ke is given by the ratio *8/9* of the chord, i.e. the *sounding part* of the chord, for Ke, is the *8/9* of the chord. So, *1/9* of the chord is the *non-sounding* part of the chord, i.e. from the small bridge (on the neck) to the fret. So, this is a distance and has the dimension of length. I.e. the full writing of the ratio should be: *[(1/9)L] : [(1/12)L]*, where L is the length of the chord. And now what is this Ke:Zo?- It is not Ke Zo at all. The Elasson tonos is given, according to Chrysanthos, by the ratio *11/12* (a ratio taken isolated from the "omalon diatonon" of Ptolemaios). So, the sounding part of the chord for an Elasson is *(11/12)L* and *(1/12)L* is the non sounding part from the small bridge to the fret. Which fret? A fret for Zo? - No! If the fret for Ke was on *(1/9)L* from the *small bridge*, the *(1/12)L* is a fret closer to the small bridge, producing not Zo but a lowered Ke! And what is compared here are not the acoustical intervals for meizon and elasson tonos, but the distances of the frets from the small bridge, the frets being put *on this special position* and not elsewhere on the chord. So, if *M=(1/9)L cm* and *E=(1/12)L cm*, Chrysanthos calculates in fact the ratio *M/E= [(1/9)L cm]/[(1/12)L cm]=12/9*. So, if *L=12*9 cm=108 cm* (which roughly represents the chord length of tanbur), then *M=(1/9)*108 cm=12 cm* and *E=(1/12)L cm=(1/12)*108 cm=9 cm*. Or to repeat Chrysanthos calculation in a fuller and more rigorous form (I put the units of length, say cm, where needed):

M cm/ E cm= [(1/9)L cm]/[(1/12)L cm].

Now, Chysanthos *presupposes* (see p. 97-98) the number 12 for the meizon tonos, which, according to what we told above, should be the length of 12 cm, otherwise the dimensions in the calculation will be wrong (Chr. is probably inspired by the ancient greek theory to give this number because he does not give a proof for it). So, with $E=\chi$, the above equation becomes:

$$M\ cm/\ E\ cm= [(4/36)L\ cm]/[(3/36)L\ cm]=(4/36)/(3/36)=>$$
$$12\ cm/(4/36)=\chi\ cm/(3/36)=>$$
$$(4/36)/12\ cm=/3/36)/\chi\ cm =>$$
$$4/(12*36)\ cm)=3/(36\chi\ cm)=>$$
$$4*36\chi\ cm=12*36*3\ cm =>$$
$$\chi=9\ cm.$$

Or more simply:

$$M\ cm/\ E\ cm= [(1/9)L\ cm]/[(1/12)L\ cm]=12/9 =>$$
$$E=\chi=9\ cm.$$

This is what I called "first method of Chrysanthos" and he stops with it here, having calculated kommata for Meizon and Elasson, kommata having the dimension of length. He does not calculate kommata for elahistos with this method. If we continue with this method to calculate it, we will have a still lower fret having a distance *[1-(81/88)]L= [(7/88)]L* from the small bridge. And on the previous chord of *L=108 cm* this is *(7/88)*108 cm=8.59 cm*. (This can been shown also in another way, by applying the above method of Chrysanthos and calculating the ratio M/e or the ratio E/e, where e is the distance of the fret for the elahistos, which produces a lower Ke, from the small bridge)

So, with the first method of Chrysanthos, the 4chord Di-Ni should be:

Di 12 Ke 9 Zo 8.59 Ni

For the calculation of the kommata for elahistos, Chr. changes his method. He writes on pp 26-27, note b.: "*When the whole chord is supposed 27, to Di corresponds the fraction 27/27, ie. 1, to Ke (corresponds) 24/27, ie 8/9 ,to Zo (corresponds) 22/27 and to Ni (corresponds) ¾".*

First, can this "27" be a pure number?-No, it is a length, say 27 cm, and the rest ratios represent frets having distances *from the big bridge*, the bridge on the soundboard, *(8/9)L*, *(22/27)L* and *(3/4)L*. Now, these frets are placed on their right places, i.e. Zo higher than Ke and Ni higher than Zo. And Chrysanthos continues to calculate the kommata for elahistos, writing: "*consequently, to the interval Zo Ni (corresponds) 7/108, because (1/4)-(5/27)=(27/108)-(20/108)=7/108, so Di Ke: Zo Ni::(1/9):(7/108), but also (1/9):12::(7/12*9):x. Consequently (1/9)x=(12*7)/(12*9); so, x=(7*9)/(9*1)=63/9=7*".

But what is now this Di Ke? Is it the ratio representing the meizon tonos? -No, it is the distance between the fret of Ke and the small bridge (=the "fret" for Di) equal to *(1/9)L* as in the first method. And what is now the "interval Zo Ni"? Is it the ratio representing the elahistos?-No, it is the *distance between the frets of Zo and Ni*. Because the fret of Ni lies on *[1-(3/4)]L=(1/4)L* from the small bridge and the fret of Zo lies on *[1-(22/27)]L=(5/27)L* from the small bridge, so *(1/4)-(5/27)*, or more fully *[(1/4)-(5/27)]L =(7/108)L* is the distance between the frets of Zo and Ni. It's not a ratio at all. It's a distance *on this special place on the chord* (because an elahistos will have a different distance at a higher place of the chord). So the calculations of Chrysanthos in full form should be (I include also the units of length, say cm, where needed): let *M΄cm* be the distance Di Ke and χ *cm* be the distance Zo Ni (we showed

that the Zo Ni included in Chrysanthos calculation is a length). Then the above calculation is written:

$$M'cm/\chi\ cm=(1/9)L\ cm/(7/108)L\ cm=(1/9)/(7/108) \rightarrow$$
$$M'cm/(1/9)=\chi\ cm/(7/108).$$

Now, Chrysanthos takes a number for the meizon tonos from his first calculation which was made with a different method, so mixing up them. He takes the number 12 k. but as we showed this must be 12 cm (this is a mixing, but, as can be shown, we have again 12 cm for the meizon if we apply the second method). So,

$$M'cm/(1/9)=\chi\ cm/(7/108)\rightarrow$$
$$(1/9)/M'cm=(7/108)/\chi\ cm\rightarrow$$
$$(putting\ M'=12\ cm)(1/9)\chi\ cm=12\ cm*(7/108)=(12*7\ cm)/(12*9) \rightarrow$$
$$\chi=7\ cm.$$

Chrysanthos stops here with his second method. As a final result of his calculations he gives the numbers 12-9-7 of which the first is given by definition (though it can be a natural result of the kind of his calculation, if we take the chord having the length of 108 cm), the second is calculated through his first method and the third through his second method.

But let us proceed with his second method to calculate the kommata for the Elasson tonos. According to this method, the “magnitude of Elasson tonos” must be defined as the distance between the frets of Zo and Ni. This is $(8/9)L-(22/27)L=(24/27)L-(22/27)L=(2/27)$ or, to use distances from the small bridge as Chrysanthos does (see above), $(5/27)L-(1/9)L= (5/27)L-(3/27)L=(2/27)L$. Let’s call this distance x. Then, calculating in the way of Chrysanthos:

$$M /x =(1/9)L/(2/27)L=3/2=>$$
$$x=2M/3.$$

And if $M=12\ cm$, then

$$x=(2*12\ cm)/3=> x=8\ cm.$$

So, with the second method of Chrysanthos, the 4chord Di-Ni should be:

Di 12 Ke 8 Zo 7 Ni

Let’s now proceed a step further and try to calculate the Meizon tonos Ni Pa, using the second method od Chrysanthos. This must be defined as the distance between the corresponding frets which is $(2/3)L-(3/4)L=(1/12)L$ or $(1/3)L-(1/4)L=(1/12)L$. So, if M is the Meizon tonos Di Ke=12 cm and M΄ is the Meizon tonos Ni Pa (as defined in the second method of Chrysanthos), then:

$$M/M'=(1/9)L/(1/12)L=12/9=4/3=>$$
$$M'=3M/4=3*12\ cm/4=>$$
$$M=9\ cm!$$

So, two meizones tonoi (acoustically equal intervals) have not the same number of kommata. After this, the kommata as defined by Chrysanthos cannot be added to produce the number of kommata of a full scale. It has no meaning to say that the octave has 68 (=12+9+7+12+12+9+7) or 66 (=12+8+7+12+12+8+7) kommata. Nor has it any meaning to divide the octave in 68 or 66 acoustically equal kommata by using the method of logarithms described at the beginning of this article. The use of these kommata, either as given by Chrysanthos or as divisions of the octave in 68 or 66 acoustically equal parts, should be discarded. Nevertheless, the ratios given by Chrysanthos can still be used and be transformed to correctly calculated kommata through the logarithmic method described at the beginning of this article. If we divide *by definition* the Meizon tonos in 12 (acoustically equal) kommata, then the ratios used by Chrysanthos give the 4chord 12-9-8.5 which can be approximated by 12-9-9 to give an octave of 72 kommata as usually[1].

[1] If we divide the *octave* in 72 kommata, then the Meizon tonos will be 12.23 k or, in other words, an interval of 12 k will be slightly smaller than a ‘true’ Meizon tonos; it will be a tempered whole tone. I use the method of dividing the *Meizon tonos* in *12* k as a little more accurate and approximating the ‘true’ intervals (‘true’ in the sense that they are expressed through *ratios*) with simpler numbers.

Note: The symbol * has been used for multiplication.